

How To Satisfy The Never-Satisfied Student

For each of the following, what we are taught to present satisfies most students, but – on closer inspection – there are more ‘holes’ in what we teach (or at least I teach!) than what we might think.

I hope that there will be the opportunity for you to discuss this and share your thoughts, but here is a non-definitive list of prompts, even with a ‘mark the student’s work’ and ‘try this integral’ exercise at the end.

At the end of this session I hope to present my thoughts on any of what is listed here, but also to give the opportunity for people to add to their own ideas.

Pre-A Level

1. Do you derive the quadratic formula or just state it?

Maths A Level

2. Factor Theorem

Students are quickly satisfied that if $(x - a)$ is a factor of a polynomial, $f(x)$, then this means that you can write $f(x) = (x - a)g(x)$, where $g(x)$ is some polynomial, and so $f(a) = 0 \times g(a) = 0$.

However, do you ever get challenged on the *converse* of this theorem, that if $f(a) = 0$, for some polynomial $f(x)$ and real number a , then $(x - a)$ *must* be a linear factor of $f(x)$? If so, what do you say?

3. Partial Fractions

Students just get taught the three types of partial fractions and become familiar with what to do, but there is also a lot of trust that partial fractions will always ‘work’. How do we know there aren’t cases where it doesn’t work?

If we put this to one side, students can be unhappy about substituting $x = a$ when $(x - a)$ is a factor of the denominator of the algebraic fraction? How do you satisfy them?

4. Generalised Binomial Theorem

How do you explain why $(1 + x)^n$ is valid for $|x| < 1$?

(As an aside, does anyone else feel that students are realising that the Generalised Binomial Theorem can be used even when the power is a positive integer and so they just use this all the time, but then they lose a lot of the

understanding of Pascal's triangle and binomial coefficients?)

5. Exponentials

There is so much beauty contained in the number e , does anyone else feel that we are selling the students short if all we show them is that $y = a^x$ is its own gradient function precisely when $a = e$? Do any of you have ways of introducing more of the properties of e ?

6. Small angle approximations

How do you explain these?

■ **When θ is small and measured in radians:**

- $\sin \theta \approx \theta$
- $\tan \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$

7. Differentiation

How do you explain the Chain Rule? The Product Rule?

8. Definite Integration

Explaining indefinite integration as the reverse of differentiation (or noitaitnereffid, as I like to call it) feels quite natural, but how do you go about explaining how definite integration leads to giving the signed area between a curve and the x -axis?

Further Maths A Level

9. Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

Are you like me and you have shown how this is true for quadratic equations by using the quadratic formula and have only half justified beyond there? Is there more we could do to convince them that this holds for *all* polynomial equations with real coefficients?

10. Invariant lines.

You are taught to find invariant lines of the form $y = mx + c$, but it should be addressed that not all straight lines can be expressed in this form... so might there be others?

11. Second order differential equations.

How do you explain that the most general solution to a second order differential equation has two unknowns? It is clear when the equation is of the form $\frac{d^2y}{dx^2} = f(x)$, but less obvious when it is of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.

When you show how the general solution of the of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, when the auxiliary equation has complex roots, $\alpha \pm i\beta$, is $y = e^{\alpha x}(C \cos \beta x + D \sin \beta x)$, how do you explain the appearance/disappearance of i ? After all, in the case of Simple Harmonic Motion, for example, the solution is entirely based in the 'real world'. Talking of which, do you have a concrete way of explaining the three cases for damped SHM?

Finally, for non-homogeneous second order differential equations how do you justify using the trial function $y = xf(x)$ when $f(x)$ is already part of the complementary function? Come to think of it, how do you explain why the general solution of a homogeneous second order differential equation is $y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$ when the auxiliary equation has repeated root λ_1 ?

12. Maclaurin Series

Do any of you justify the 'valid for' part of the Maclaurin Series that students are told to just use blindly from the formula book? If so, how do you do it?

13. Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

And now for something completely different!

14. This old chestnut you have probably seen and shared before, but I wanted to briefly point this out in case you haven't come across it before. I am not aware of it being addressed in any big way in A Level text books, but this is an unresearched educated guess.

What is the integral, $\int \frac{1}{2x} dx$?

One way would be to either integrate by inspection (reverse Chain Rule, or eluR niahC, as I like to call it) or use the substitution $u = 2x$ to get:

$$\int \frac{1}{2x} dx = \frac{1}{2} \ln|2x| + c$$

Another way would be to just take the $\frac{1}{2}$ outside the integral:

$$\int \frac{1}{2x} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln|x| + c$$

Which one is correct? They cannot both be correct, can they?!

15. From the Formula Book for EdExcel:

Integration (+ constant; $a > 0$ where relevant)

$f(x)$	$\int f(x) dx$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right), \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right), \ln\{x + \sqrt{x^2 + a^2}\}$

Why do you think that I think this is misleading?!

16. Below is a student's working for an A Level internal Maths exam this summer. She seems to do some good work, yet she does not get the answer in the question. Can you figure out what has happened?

15. The number of mice on an island is being monitored.

When monitoring began there were 4000 mice on the island.

In a simple model, the number of mice x , in thousands, is modelled by the equation

$$x = \frac{k(3t+2)}{4t+5}$$

where k is a constant and t is the number of years after monitoring began.

(a) Find a complete equation for this model. (2)

(b) Hence calculate the long-term population of mice predicted by this model. (1)

In a second model, the number of mice x , in thousands, is modelled by the differential equation

$$4 \frac{dx}{dt} = 6x - x^2$$

where t is the number of years after monitoring began.

(c) Write $\frac{4}{6x - x^2}$ in partial fraction form. (3)

(d) Solve the differential equation with the given initial condition to show that

$$x = \frac{12e^{\frac{3}{2}t}}{1 + 2e^{\frac{3}{2}t}} \quad ?$$

(e) Hence find the long-term population of mice predicted by this **second** model. (5)

(1)

a) sub. $t = 0$,
 $\frac{k(2)}{5} = 4$
 $2k = 20$
 $k = 10$
 $x = \frac{10(3t+2)}{4t+5}$

b) 7500

Question 15 continued

$$c) \quad 6X - X^2 = X(6-X)$$

$$A(6-X) + BX = 4$$

$$B-A=0$$

$$6A=4$$

$$A = \frac{2}{3}$$

$$\frac{\frac{2}{3}}{X} + \frac{\frac{2}{3}}{6-X}$$

$$= \frac{2}{3X} + \frac{2}{3X-18}$$

$$d) \quad \int \frac{2}{3X} dx - \int \frac{2}{3X-18} dx = \int 1 dt$$

$$\frac{2}{3} \int X^{-1} dx - \frac{2}{3} \int (X-6)^{-1} dx = t$$

$$\frac{2}{3} \ln|X| - \frac{2}{3} \ln|X-6| + C = t$$

$$\frac{2}{3} \ln|4| - \frac{2}{3} \ln|-2| + C = 0$$

$$\frac{2}{3} \ln(2) + C = 0$$

$$C = -\frac{2}{3} \ln(2) \quad \frac{2}{3} (\ln(X) - \ln(X-6) - \ln(2)) = t$$

$$\frac{2}{3} \left(\ln \left(\frac{X}{2X-12} \right) \right) = t$$

$$\ln \left(\frac{X}{2X-12} \right) = \frac{3}{2} t$$

$$\frac{X}{2X-12} = e^{\frac{3}{2} t}$$

$$X = 2X e^{\frac{3}{2} t} - 12 e^{\frac{3}{2} t}$$

$$X(1 - 2e^{\frac{3}{2} t}) = -12 e^{\frac{3}{2} t}$$

$$X = \frac{12 e^{\frac{3}{2} t}}{2e^{\frac{3}{2} t} - 1}$$

$$e) \quad 6000$$

17. Here is an integral for you to try. After you have had a go, we can discuss our thoughts on this at the end of the session.

Use the substitution $u^2 = \sin x$ to evaluate the definite integral $\int_0^\pi \sqrt{1 - \sin x} dx$.